Math 332 * Victor Matveev
Final Exam preparation list

1) Complex numbers:

1. Cartesian representation, addition/subtraction, division ($1/z=\bar{z}/|z|^2$), complex conjugation.
2. Complex exponential and Euler equation
3. Polar representation of complex numbers: branches of argument
$$z = |z| \exp\{ i \arg z \} = |z| \exp\{ i \operatorname{Arg} z + i\, 2\pi k \}$$
4. Properties of $|z|$ and $\bar{z}$, triangle inequalities
$$|z_1 z_2| = |z_1||z_2|; \quad |z_1/z_2| = |z_1|/|z_2|; \quad |\bar{z}| = |z|$$
$$\big||z_1| - |z_2|\big| \le |z_1 \pm z_2| \le |z_1| + |z_2|$$
5. Complex roots
6. Sets in the plane (review lines and circles, $z = z_0 + r \exp(i\,t)$)

2) Functions of complex variable:

1. Function as a Mapping
2. Limits and Continuity
3. Analyticity: $f(z)$ is analytic at z_0 if its derivative exists there, as defined by a 2D limit
$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
4. Cauchy-Riemann equations hold if the function $(u + i\,v)$ is analytic : $u_x = -v_y$, $u_y = -v_x$
5. Harmonic functions and harmonic conjugates
6. Solving Laplace's equation with Dirichlet boundary conditions

3) Elementary functions

1. Polynomials and Rational functions: fundamental theorem of algebra, polynomial deflation, zeros, poles, partial fractions
2. Complex exponential, trigonometric, hyperbolic functions
$$\exp z = \exp(x)\exp(i\,y) = \exp(x)\,(\cos y + i \sin y)$$
$$\sin z = \sin x \cosh y + i \cos x \sinh y$$
$$\cos z = \cos x \cosh y - i \sin x \sinh y$$
3. Logarithmic function: branches and branch cuts
$$\log z = \log\{ |z| \exp(i \arg z) \} = \operatorname{Log}|z| + i \arg z = \operatorname{Log}|z| + i\,\{\operatorname{Arg} z + 2\pi k\}$$
4. Complex powers, inverse trig and inverse hyperbolic functions
$$z^w = \exp(w \log z)$$
$\sin^{-1}(z) = -i \log \{i\,z + (1 - z^2)^{1/2}\}$ (Derive, don't memorize)
$\cos^{-1}(z) = -i \log \{ z + (z^2-1)^{1/2}\}$ (Derive, don't memorize)
$\tan^{-1}(z) = i/2 \log \{ (1- i\,z) / (1 + i\,z) \}$ (Derive, don't memorize)

4) Contour integral:

1. Smooth arcs, simple closed curves and their parametrization; a contour is a sequence of smooth curves

2. Contour integral calculation methods:

   i. Limit of a Riemann sum: $\lim_{\max|\Delta z_k|\to 0} \sum_{k=1}^{N} f(z_k^*)\Delta z_k$

   ii. Contour parameterization: $\int f(z)\, dz = \int f(z(t))\, z'(t)\, dt$

   iii. Antiderivative ($\int f(z)\, dz = F(z_{\text{end}}) - F(z_{\text{start}})$)

   iv. Changing contour of integration (see Cauchy integral theorem below)

   v. Some loop integrals equal zero (see Cauchy integral theorem below)

3. Important integral (derive using $z = R \exp(i\, t)$): $\oint_{|z-z_0|=R} \frac{dz}{(z-z_0)^n} = \begin{cases} 0, & n \neq 1 \\ 2\pi i, & n = 1 \end{cases}$

4. Calculating upper bounds on integral modulus: $\left| \int_\gamma f(z)dz \right| \le \max_{z\subset\gamma} |f(z)|\, \ell(\gamma) \equiv M \cdot L$

5. **Theorem:** if $f(z)$ is continuous in domain D, the following statements are equivalent:
   (a) $\exists F(z) \,|\, F'(z) = f(z)$ (b) $\oint_{\forall\gamma\subset D} f(z)dz = 0$ (c) $\int_{\gamma_{AB}} f(z)\,dz = \int_{\gamma'_{AB}} f(z)\,dz$

6. **Cauchy integral theorem:**

   If $f(z)$ is **analytic** in a **simply-connected** domain D, the above three properties (a,b,c) hold.

   - Corollary 1: if a function is analytic between two simple contours with same endpoints or between two simple closed curves, the two contour integrals are equal.
   - Corollary 1*: if there is a continuous deformation of one contour into another (without crossing non-analyticities, with endpoints fixed), the two integrals are equal.

7. Corollary of above two theorems: Loop integral is zero if either of the following is true:

   (1) $f(z)$ is analytic **inside and on** the loop
   (2) $f(z)$ has a continuous anti-derivative **on** the loop (Example: $1/z^2$)

8. **Cauchy Integral Formula:**

   If $f(z)$ is analytic in D and z_0 is inside simple closed contour γ lying in D, then $\boxed{f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)dz}{z-z_0}}$;

   $$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)dz}{(z-z_0)^{n+1}}}$$

   Corollary: bounds on analytic functions: $\left|f^{(n)}(z_0)\right| \le \frac{n! \max_{|z|=R} |f(z)|}{R^n}$

   Corollary: analytic functions only reach their max modulus on the boundary of a domain. Functions analytic on unbounded domains are unbounded there.

5) Series representation of analytic functions

1. If a function is analytic at z_o, it has a **Taylor series** representation in a neighborhood of z_o:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \text{ where } c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}, \text{ contour C contains } z_0$$

T.S. converges in $|z-z_0|<R$, converges uniformly in $|z-z_0| \le R' < R$, and diverges in $|z-z_0| > R$

2. If a function is analytic in $r<|z-z_0|<R$, it has a **Laurent series** representation there:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n + \sum_{n=1}^{\infty} c_{-n} (z - z_0)^{-n}$$

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}, \text{ where C is inside the ring and contains } z_0$$

The first term (positive-power series) converges in $|z-z_0|<R$, while the second term (principal part) converges in $|z-z_0|>r$. Laurent series diverges outside of the ring $r<|z-z_0|<R$

3. Convergence radius: $R = \lim_{j\to\infty} | c_j / c_{j+1} |$ (from ratio test) or $R = 1 / \limsup_{j\to\infty} \sqrt[j]{| c_j |}$ (from root test)

4. Use term-by-term operations to derive Taylor and Laurent series, avoiding explicit differentiation or integration. Use a simple shift to expand around non-zero z_0.

5. Remember important series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $Log(1+z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}$, ...

6. If a function has an isolated singularity, it has a Laurent series expansion centered at that point. Isolated singularities are:
   (1) Removable singularity: $c_{-n} = 0$ for all $n > 0$ (Laurent series = Taylor series)
   (2) Pole of order m: $c_{-n} = 0$ for all $n > m$. Function modulus is unbounded near the pole.
   (3) Essential Singularity: infinitely many non-zero c_{-n} (where $n > 0$). Function assumes every possible value with possibly one exception infinitely many times in any neighborhood of E.S.

7. A function has **no** series representation in any neighborhood of non-isolated singularity such as a branch point, branch cut, or an accumulation point (e.g. $1/\sin(1/z)$ at $z_0=0$)

8. Alternative definitions of a zero: z_o is a zero of order m of $f(z)$ if the function is analytic there and:
   (1) $f^{(n)}(z_0) = 0$ for $n < m$, but $f^{(m)}(z_0) \neq 0$
   (2) $f(z) = (z - z_0)^m g(z)$, where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_o
   (3) $f(z) = 0 + 0 + ... + 0 + c_m (z - z_0)^m + c_{m+1} (z - z_0)^{m+1} + c_{m+2} (z - z_0)^{m+2} + ...$, where $c_m \neq 0$

9. Alternative definitions of a pole: z_o is a pole of order m of $f(z)$ if:
   (1) $1/f(z)$ has a zero of order m at z_o
   (2) $f(z) = \frac{g(z)}{(z - z_0)^m}$, where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_o
   (3) $f(z) = \frac{c_{-m}}{(z - z_0)^m} + \frac{c_{-m+1}}{(z - z_0)^{m-1}} + ...$, where $c_{-m} \neq 0$

6) Cauchy's Residue Theorem and applications:

1. Term-by-term integration of a Laurent series gives:
   $\oint_C f(z)\,dz = 2\pi i\; a_{-1}$, where C contains a single isolated singularity z_0,
   a_{-1} is called the ***residue*** of function $f(z)$ at z_0

2. Therefore, if $f(z)$ is analytic inside C except for the isolated singularities z_i, then:
   $$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f; z_j)}$$

3. Residue calculation methods:

   1) $\mathrm{Res}(f; z_0) = a_{-1}$ (definition; works for all isolated singularities)

   2) Pole of order m: just count the powers, and you get the Cauchy Integral Formula:
   $$\mathrm{Res}\left(\frac{g(z)}{(z-z_0)^m}; z_0\right) = c_{-1}^{f} = c_{m-1}^{g} = \frac{1}{(m-1)!}\left.\frac{d^{m-1}g(z)}{dz^{m-1}}\right|_{z\to z_0} = \frac{1}{(m-1)!}\left.\frac{d^{m-1}\left(f(z)(z-z_0)^m\right)}{dz^{m-1}}\right|_{z\to z_0}$$

   3) Simple pole: $f(z)=p(z)/q(z)$, where $p(z_0)\neq 0$, $q(z_0)=0$:
   $$\mathrm{Res}\left(\frac{p(z)}{q(z)}; z_0\right) = \frac{p(z_0)}{q'(z_0)}$$

4. Special integrals taken using residue method:

   1) Trigonometric integrals over a whole period: make a substitution $z = \exp(i\,t)$
   2) Improper integrals over rational functions from $-\infty$ to $+\infty$: complete the integration contour in the top or bottom half-plane
   3) Improper integrals involving trig functions – replace trig functions with complex exponentials; complete the integral in the top or bottom half-plane; use the Jordan's Lemma.
   4) Poles on the real axis – use indented contour. Integral over half a circle surrounding a simple pole is equal $2\pi i$ times half the residue, in the limit of circle radius approaching zero
   5) Integrals involving multi-valued functions – integrate over the branch cut
   6) Improper integrals of rational functions from 0 to ∞ which are neither even nor odd – multiply integrand by zero branch of log z; integrate over the branch cut.

Jordan's Lemma:

$\oint_{C_\rho} R(z)e^{imz}\,dz \le \frac{\pi M_R}{m}$ where $M_R = \max_{z\subset C_\rho} |R(z)|$, and C_ρ is a semi-circle in the top half-plane

Properties of functions $f(z)$ analytic in domain D:

1) $f(z)$ can be expressed as a function of $z = x+i\,y$ only
2) df/dz exists in D (definition of analyticity)
3) All higher-order derivatives also exist in D (given by the C.I.F.)
4) $f(z)$ has a Taylor series representation in a neighborhood of any point in D
5) Cauchy-Riemann identities hold ($u_x = v_y,\ u_y = -v_x$)
6) u=Re(f) and v=Im(f) are harmonic in D
7) $f(z)$ is uniquely determined by its values over any single curve or open set in D.
   [C.I.F. tells us how to determine $f(z)$ from its values along a loop around z]
8) $f(z)$ at the center of any circle in D equals it average over the entire circle
9) $|f(z)|$ can only reach its maximum on the boundary of D
10) If D is unbounded, then $f(z)$ is unbounded
11) If D is simply connected, then Cauchy Integral Theorem applies:
    a) All loop integrals of $f(z)$ in D are zero, and all open contour integrals are path independent
    b) $f(z)$ has an antiderivative in D